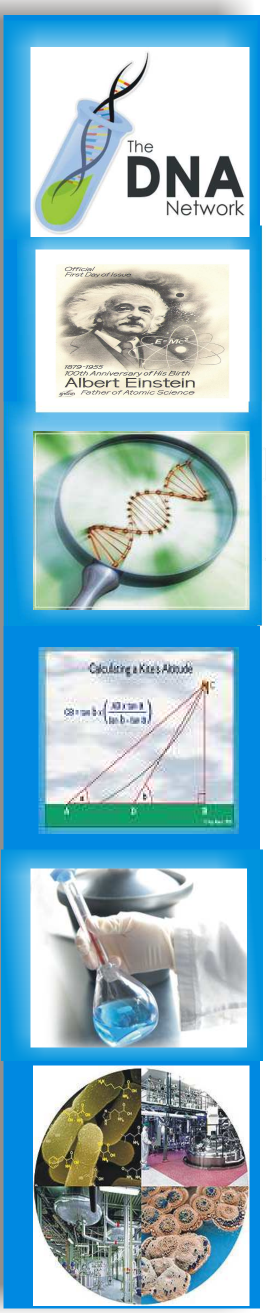

IIT JAM - MATHEMATICS

SAMPLE THEORY

- AREA OF THE CURVE
- GROUP AND GROUP OF PERMUTATION
- UNIFORM CONVERGENCE AND DIFFERENTIATION

VPM CLASSES

For IIT-JAM, JNU, GATE, NET, NIMCET and Other Entrance Exams

1-C-8, Sheela Chowdhary Road, Talwandi, Kota (Raj.) Tel No. 0744-2429714

Web Site www.vpmclasses.com E-mail-vpmclasses@yahoo.com

Area by Double Integration

Let the area ABCD be divided into sub-areas by drawing lines parallel to x and y-axis respectively such that the distance between two adjoining lines drawn parallel to y-axis be δx and those drawn parallel to x-axis be δy.

Let P(x, y) and Q(x + δx, y + δy) be two neighbouring points on the curve AD whose equation is y = f(x) as in case (a). PN and QM are the ordinates at P and Q respectively. Then the area of the element, shown by shaded lines in adjoining figure is $\delta x \delta y$.

Consequently the area of the strip PNMQ = $\int_{y=0}^{f(x)} dxdy$, where y = f(x) is the equation of AD.

$\therefore$ The required area ABCD = $\int_{x=a}^{b}\int_{y=0}^{f(x)} dxdy$.

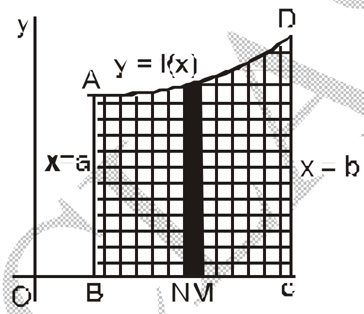

Fig.1

In a similar way, we can prove that the area bounded by the curve x = f(y), the y-axis and the abscissa at y = a and y = b is given by

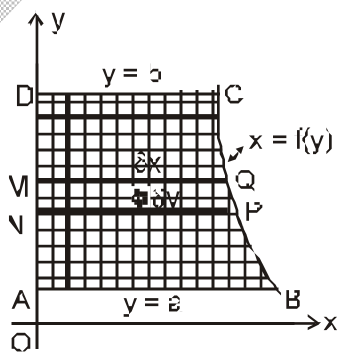

Fig.2

If we are to find the area bounded by the two curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a$ and $x = b$ i.e. the area ABCD in the figure below then the required area = $\int_{x=a}^{b}\int_{y=f_2(x)}^{f_1(x)} dxdy.$

Ex. Find the area of the region bounded by the parabolas $y = x^2$ and $y = 4 - x^2$.

Sol. $x^2 = y$ represents a parabola whose vertex is at (0, 0) and $y = 4 - x^2$ represents a parabola whose vertex is at (0, 4).

Solving the two equations we get $y = 4 - y$ or $2y = 4$ or $y = 2$

$\therefore x^2 = 2$ or $x = \pm\sqrt{2}$

$\therefore$ The two parabolas intersect at $A\left(\sqrt{2}, 2\right)$ and $B\left(-\sqrt{2}, 2\right)$.

$\therefore$ Required area = 2(area OACO) = 2[area OADO + area DACD] = $2\left[\int_{y=0}^{2} xdy + \int_{y=2}^{4} xdy\right]$,

(where the first integral is for $x^2 = y$ and second for $y = 4 - x^2$)

$$= 2\left[\int_0^2 \sqrt{y}dy + \int_2^4 \sqrt{(4-y)}dy\right] = 2\left[\left(\frac{2}{3}y^{\frac{3}{2}}\right)_0^2 - \left\{\frac{2}{3}(4-y)^{\frac{3}{2}}\right\}_2^4\right]$$

$$= 2\left[\left(\frac{4}{3}\right)\sqrt{2} + \left(\frac{2}{3}\right)(2)^{\frac{3}{2}}\right] = \left(\frac{16}{3}\right)\sqrt{2}.$$ **Ans.**

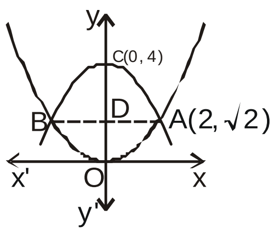

Fig.3

Area of curve given by polar equation

(a) Single Integration

The area bounded by the curve $r = f(\theta)$, where $f(\theta)$ is a single value continuous function of θ in the domain (α, β) and the radii vectors $\theta = \alpha$ and $\theta = \beta$ is equal to $\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$, $(\alpha < \beta)$

Proof. Let O be the pole, OX the initial line and AB be the portion of the arc of the curve $r = f(\theta)$ between the radii vectors $\theta = \alpha$ and $\theta = \beta$.

Let $P(r, \theta)$ be any point on the curve between A and B. Let $Q(r + \delta r, \theta + \delta\theta)$ be a point neighbouring to P. Join OP and OQ. With OP = r as radius and O as centre describe an arc PN of the circle meeting OQ in N. Similarly with O as centre and OQ = r + δr as radius draw another arc QM of circle meeting OP produced in M.

Let the sectorial areas OAP and OAQ be denoted by S and S + δS respectively.

Then the area OPQ = $(S + \delta S) - S = \delta S$.

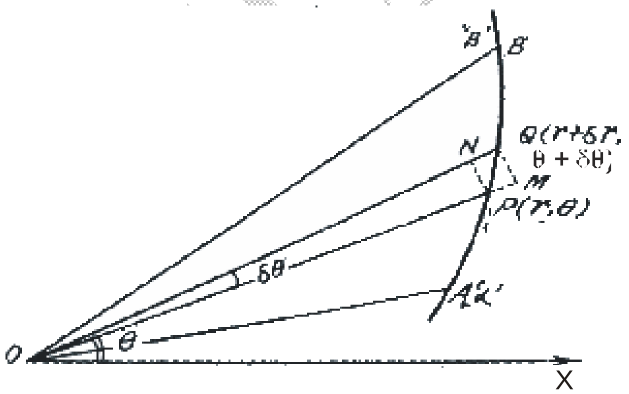

Fig. 4

Also as OP = r, OQ = r + δr and $\angle POQ = \delta\theta$ so sectorial area OPN = $\frac{1}{2}r^2\delta\theta$ and sectorial area OQM = $\frac{1}{2}(r+\delta r)^2\delta\theta$. Now area OPQ lies between area OPN and area OQM i.e. Area δS lies between $\frac{1}{2}r^2\delta\theta$ and $\frac{1}{2}(r+\delta r)^2\delta\theta$. i.e. $\left(\frac{\delta S}{\delta\theta}\right)$ lies between $\frac{1}{2}r^2$ and $\frac{1}{2}(r+\delta r)^2$

$\therefore$ In the limit as $\delta\theta \to 0$, we have $\left(\frac{dS}{d\theta}\right) = \frac{1}{2}r^2$ or $\frac{1}{2}r^2 d\theta = dS$

Integrating $\frac{1}{2}\int_\alpha^\beta r^2 d\theta = [\text{area } S]_\alpha^\beta$, = (Area S when $\theta = \beta$) – (Area S when $\theta = \alpha$) = (Area AOB) – (0) = Area AOB.

Hence, required area AOB = $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$.

(b) Double Integration

The area bounded by the curve r = f(θ), where f(θ) is single valued function of θ in the domain (α, β) and radii vectors $\theta = \alpha$ and $\theta = \beta$ is $\int_{\theta=\alpha}^{\beta}\int_{r=0}^{f(\theta)} r\,d\theta\,dr$

Ex. Find by double integration the area lying inside the cardioid r = a(1 + cosθ) and out side the circle r = a.

Sol. Required area = area ABCDA = 2(area ABDA) =

$$= a^2\int_0^{\frac{\pi}{2}}[(1+\cos\theta)^2 - 1]d\theta = a^2\int_0^{\frac{\pi}{2}}(\cos^2\theta + 2\cos\theta)d\theta = a^2\left[\frac{1}{2}\cdot\frac{\pi}{2} + 2(\sin\theta)_0^{\frac{\pi}{2}}\right] = a^2\left[\frac{\pi}{4} + 2\right]$$

$= \frac{1}{4}a^2(\pi + 8)$ **Ans.**

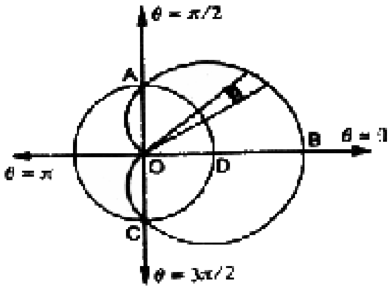

Fig 5

GROUP

Binary Operation

Let S be a non-empty set. Any function from S x S to S is called binary operation. i.e.

if $o : s \times s \to s$ defined as $\circ\ (a,b) = a \circ b \in S,\ \forall a, \in S$, then is binary operation.

Mathematical Structure

Let S be a non empty set. Let be an operation on S then (S,) is a mathematical structure.

Grouped (Quasi - group)

Mathematical structure (S,) is said to be grouped, if is binary operation i.e., .

$\forall a,b \in S \Rightarrow a \circ b \in S$

Semi group

A group (S, $\circ$), is semi group if it is associative i.e.,

Monoid $(a \circ b) \circ c = a \circ (b \circ c), \forall a,b,c \in S$

If identity element $e \in S$ exist in a semi group (S,), then it is monoid, i.e., $\forall a \in S, \exists e \in S : a \circ e = a = e \circ a$

Group

If inverse element exists for every element in a monoid (S,), then it is a group, i.e., $\forall a \in S$, $\exists a^{-1} \in S : : a^{-1} = e\ = a^{-1} \circ a$

Commutative group (Abelian Group)

A group (S,), is a commutative group, if . $\forall a,b \in S, a \circ b = b \circ a$

The Different Groups

	Quasi group	Semi group	Monoid	Group	Abelian Group
Clouser	✓	✓	✓	✓	✓
Associative	-	✓	✓	✓	✓
Existence of Identify	-	-	✓	✓	✓
Existence of inverse	-	-	-	✓	✓
Commutative	-	-	-	-	✓

Table - 1

GROUP

Definition

Let $G \neq \phi$ be a set. Let be an operation defined in G, then mathematical structure $(G,\circ)$ will be group if it satisfies.

(i) Closure law : $\forall a,b \in G \Rightarrow a \circ b \in G$

(ii) Associative law: $(a \circ b) \circ c = a \circ (b \circ c), \forall a,b,c \in G$

(iii) Existence of identity: $\forall a \in G, \exists e \in G : a \circ e = a = e \circ a$

(iv) Existence of inverse: $\forall a \in G, \exists a^{-1} \in G : a \circ a^{-1} = e = a^{-1} \circ a$

Results

- Identity element in a group is unique.
- Inverse of each element of a group is unique.
- If a,b $\in$ G, then $(ab)^{-1} = b^{-1} a^{-1}$. This law is known as reversal rule. One can generalize it as $(abc.....z)^{-1} = z^{-1}..... c^{-1}b^{-1}a^{-1}$.
- Cancellation law holds in a group. i.e. $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow c = a$
- If a,b $\in$ G, then linear equations $a \circ x = b, y \circ a = b$ have unique solutions for x, y$\in$ G

Order of Group

The number of elements in a group G, is order of group denoted by of O(G).

If (G, *) if an infinite group then it is said to be of infinite order.

Order of Element

Let G be a group. Let $a \in G$, then n is called order of element a, denoted by O(a) = n, if $a^n = e$, where n is least positive integer.

Results on Order of an Element of a Group

- The order of every element of a finite group is finite.
- If there is no positive integer n such that $a^n = e$, than order of a, o(a) is infinite or zero.
- The order of every element of a finite group is less than or equal to the order of the group. If G is a finite then $o(a) \le 0(G), a \in G$.
- The order of an element of a group is same as that of its inverse.
- Order of any integral power of an element $a \in G$ cannot exceed the order of a.
- If $a \in G$ a group ,o(a) = n and $a^m = e$, then n/m.
- If $a \in G$ is an element of order n and p is prime to n, then a^p is also of order n.
- If every element of a group except the identity element is of order two, then G is abelian.
- If every element of a group G is its own inverse, then G is abelian.

Theorem . If order of an element a of a group (G,*) is n then $a^m = e$, iff m is a multiple of n.

Proof. Let $a^m = e$

By division algorithm $m = nq + r$, $0 \le r \le n$ where $q, r \in Z$

$\therefore \quad a^m \Rightarrow a^{nq+r} = e$

$\Rightarrow a^{nq}.a^r = e$

$\Rightarrow (a^n)^q.a^r = e \quad \left[\because (a^m)^n = a^{mn}\right]$

$\Rightarrow e^q.a^r = e \quad \left[\because O(a) = n \Rightarrow a^n = e\right]$

$\Rightarrow a^r = e$

$\Rightarrow r = 0 \qquad [\therefore 0 \le r \le n]$

$\Rightarrow m = mq$

So, n/m

Conversely

Let m is multiple of n i.e. $m = nq (q \in Z)$

$m = nq \Rightarrow a^m = a^{nq} = (a^n)^q = e^q = e$

So, $a^m = e \Leftrightarrow$ m is multiple of O(a).

If a, $x \in G$ a group, then $O(a) = O(x^{-1} ax)$

Theorem . For any element a of group of G:

$O(a) = O(x^{-1}ax), \ \forall \ x \in G$

Proof. Let $a \in G, x \in G$

$(x^{-1}ax)^2 = (x^{-1}ax)(x^{-1}ax)$

$= x^{-1}(xx^{-1})ax$

$= x^{-1} aeax$

$= x^{-1}(aea) x$

$= x^{-1}a^2x$

Again consider that $(x^{-1}ax)^{n-1} = x^{-1}a^{n-1}x$, where $(n - 1) \in N$

$\Rightarrow \ (x^{-1}ax)^{n-1}(x^{-1}ax) = (x^{-1}a^{n-1}x)(x^{-1}ax)$

$\Rightarrow \ (x^{-1}ax)^n = x^{-1}a^{n-1}(xx^{-1})ax$

$= x^{-1}a^{n-1}(eax)$

$= x^{-1}(a^{n-1}a)x = x^{-1} a^n x$

By mathematical induction

$\left(x^{-1}ax\right)^n = x^{-1}a^n x, \forall\, n \in N$

now let $O(a) = n$ and $O(x^{-1}ax) = m$

$(x^{-1}ax)^n = x^{-1}a^n x = x^{-1}ex = e$

$\Rightarrow O\left(x^{-1}ax\right) \le n$

$\Rightarrow m \le n$ (1)

Again $O(x^{-1}ax) = m \Rightarrow \left(x^{-1}ax\right)^m = e$

$\Rightarrow x^{-1}a^m x = e$

$\Rightarrow x\left(x^{-1}a^m x\right)x^{-1} = xex^{-1} = e$

$\Rightarrow \left(xx^{-1}\right)a^m\left(xx^{-1}\right) = e$

$\Rightarrow ea^m e = e$

$\Rightarrow O(a) \le m$

$\Rightarrow n \le m$ (2)

By (1) and (2) $n = m$

$\Rightarrow O(a) = O\left(x^{-1}ax\right)$

If $O(a)$ is infinite then $O(x^{-1}ax)$ is also infinite.

Ex. If a,b are elements of an abelian group G, then prove that :

$(ab)^n = a^n b^n, \forall\, n \in Z$

Sol. Case (i) When $n = 0$

$(ab)^0 = e = ee$ [By the Def^n]

$= a^0 b^0$

Case (ii) When $n > 0$;

$(ab)^1 = ab = a^1 b^1$

Result is true for $n = 1$

Let result is true for $n = K$

$(ab)^k = a^k b^k$

$\Rightarrow (ab)(ab)^k = (ab)(a^k b^k)$

$\Rightarrow (ab)^{k+1} = a(ba^k)b^k$ [associativity]

$= a(a^k b)b^k$

$= (aa^k)(bb^k)$

$= a^{k+1} b^{k+1}$

By mathematical induction result is true for all integers

Case (iii) When $n < 0$ Let $n = -m$ where $m \in Z^+$

$(ab)^n = (ab)^{-m} = [(ab)^m]^{-1}$

$= (a^m b^m)^{-1}$

$= (b^m a^m)^{-1}$

$= (a^m)^{-1} (b^m)^{-1}$

$= a^{-m} b^{-m}$

$= a^n b^n$

By above conditions

G is Commutative $\Rightarrow (ab)^n = a^n b^n, \ \forall n \in Z$

Permutation

Let P be a finite set having n distinct elements. Then a one-one mapping onto itself $f : P \to P$ is called a permutation of degree n, in the finite set P is called the degree of the permutation.

Let $P = \{a_1, a_2, a_3\}$ be a finite set having n distinct elements. If $f : P \to P$ is one - one onto, then f is a permutation of degree n. Let f is a permutation of degree n.

Let $f(a_1) = b_1$, $f(a_2) = b_2$,.....$f(a_n) = b_n$ symbolically one can write it as

$f = \begin{pmatrix} a_1 & a_2 & \ldots\ldots & a_n \\ b_1 & b_2 & \ldots\ldots & b_n \end{pmatrix}$, where each element in the second row is f image of the elements of the first row.

Equality of two permutations

Two permutations f_1 and f_2 on P are said to be equal. If we have $f_1(a) = f_2(a)$.

Total number of distinct Permutations P

Let P be a finite set having n distinct elements. There shall be n! permutations of degree n, of the element in a set.

Identity Permutations

If I is a permutation of degree n such that I replace each element by itself, I is called the identity permutation of degree n.

Inverse of a Permutation

If f is a permutation of degree n defined on a finite non-empty set P. Since f is one-one onto, it is inverse able.

$$f = \begin{pmatrix} a_1 & a_2 & \ldots\ldots & a_n \\ b_1 & b_2 & \ldots\ldots & b_n \end{pmatrix} \text{then } f^{-1} = \begin{pmatrix} b_1 & b_2 & \ldots\ldots & b_n \\ a_1 & a_2 & \ldots\ldots & a_n \end{pmatrix}$$

- f^{-1} is obtained by interchanging the rows of f because $f(a_1) = b_1$. $\Rightarrow f^{-1}(b_1) = a_1$

Products or composite of permutations

If two permutations of degree n be

$$f_1 = \begin{pmatrix} a_1 & a_2 & \ldots\ldots & a_n \\ b_1 & b_2 & \ldots\ldots & b_n \end{pmatrix} \text{and } f_2 = \begin{pmatrix} b_1 & b_2 & \ldots\ldots & b_n \\ c_1 & c_2 & \ldots\ldots & c_n \end{pmatrix}$$

Then the products of these two functions is defined as

$$f_1 f_2 = \begin{pmatrix} a_1 & a_2 & \ldots\ldots & a_n \\ c_1 & c_2 & \ldots\ldots & c_n \end{pmatrix}$$

- The product $f_1 f_2$ is also a permutation of degree n.
- Product of permutations is not necessarily commutative.

Associativity of permutation. The associative law is true for the product of the permutations i.e. f, g and h are permutations, then (fg)h = f(gh)

Group of Permutations

The set of all the permutations of a given non-empty set A is denoted by S_A. Therefore if A = {a, b}, then

$$S_A = \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}$$

If A = {a, b, c}, then

$$S_A = \left\{ \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \right\}$$

It can be easily seen that

$$O(A) = n \Rightarrow O(S_A) = n!$$

Even and odd permutation

A permutation is said to be an even permutation if it can be expressed as a product of an even number of transposition.

- A permutation can not be both even or odd i.e, permutation f is expressed as a product of transposition, then the number of transposition is either always even or always odd.
- Identity permutation is always an even permutation.
- The product of two even permutation is an even permutation.
- The product of two odd permutations is an even permutation.
- A cycle of length n can be expressed as the product of n-1 permutation.
- The inverse of an even permutation is an even permutation and the inverse of an odd permutation is an odd permutation.
- Out of n! permutations on n symbols $\frac{1}{2}n!$ are even $\frac{1}{2}n!$ are odd.

Alternating group. (Group of even permutation).

On the basis of the above conclusions of the product of even and odd permutations of any set, we will show that the set of permutations is also a group.

Theorem . The set A_n of all even permutations of degree n is a group of order $\frac{1}{2}n!$ for the product of permutations.

Important Results

(i) When n = 3, $A_3 = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$

(ii) A_n is a simple group for $n \geq 5$

Every group of prime order is a simple group because such group has no proper subgroup.

(iii) The set of odd permutations of degree n is not a group because it is not closed for multiplication.

(iv) If H is a sub group of G and $N \triangleleft G$, then $H \cap N$ need not be normal in G.

For example, let

$N = A_4 = \{(1), (123), (124), (132), (134), (142), (143), (234), (243), (12)\ (34), (13)\ (24), (14)\ (23)\}$

$H = \{(1), (1234), (13)\ (24), (1432), (12)\ (34), (14)(23), (13)\ (24)\}$

This can be easily verified that

$N \triangleleft S_4$ and H is a subgroup of S_4.

But $H \cap N$ is not a normal subgroup of S_4

(v) $\frac{S_3}{A_3}$ is a commutative and cyclic group, being group of order 2 but S_3 is non abelian and not a cyclic group.

(vi). The alternating group A_n of all even permutations of degree n is a normal subgroup of the symmetric group S_n.

i.e. $A_n \triangleleft S_n$

Ex. If

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}, \ \sigma = (1\ 3\ 4)\ (5\ 6)\ (2\ 7\ 8\ 9)$$

then find $\sigma^{-1}\rho\sigma$ and by expressing the permutation ρ as the product of disjoint cycles, find whether ρ is an even permutation or odd permutation. Also find its order.

Sol. $\sigma = (1\ 3\ 4)\ (5\ 6)\ (2\ 7\ 8\ 9)$

$$= \begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 2 & 7 & 8 & 9 \\ 3 & 4 & 1 & 6 & 5 & 7 & 8 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{pmatrix}$$

$$\therefore \quad \sigma^{-1} = \begin{pmatrix} 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 3 & 6 & 5 & 2 & 7 & 8 \end{pmatrix} \quad \text{....(1)}$$

$$\text{Again } \rho\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix} \quad \text{...(2)}$$

$\sigma^{-1}\rho\sigma = \sigma^{-1}(\rho\sigma)$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 3 & 6 & 5 & 2 & 7 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \\ 8 & 9 & 5 & 2 & 6 & 3 & 1 & 4 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 5 & 2 & 6 & 3 & 1 & 4 & 7 \end{pmatrix} = (1\ 8\ 4\ 2\ 9\ 7)\ (3\ 5\ 6)$$

$$\text{Again } \rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix} = (1\ 7\ 2\ 8\ 3\ 9)\ (4\ 6\ 5)$$

$= (1\ 9)\ (1\ 3)\ (1\ 8)\ (1\ 2)\ (1\ 7)\ (4\ 5)\ (4\ 6)$

= product of 7 (odd) transpositions.

Since ρ is equal to the product of odd transpositions,

Therefore this is a odd permutation.

Finally, O(p) = L. C. M. of {O(1 7 2 8 3 9), O(4 6 5)}

= L. C. M of {6, 3} = 6

Uniform convergence of sequences

Suppose that the sequence $\{f_n(x)\}$ converges for every point x in R. It means that the function f_n tends to a definite limit as $n \to \infty$ for every x in R. This limit will be a function of x, say f. Then from the definition of a limit it follows that for every $\in > 0$, there exists a positive integer m such that

$$n \geq m \Rightarrow |f_n(x) - f(x)| < \in.$$

The integer m will depend upon x as well as $\in$ and so we may write it symbolically as m(x, $\in$). Now suppose that we keep $\in$ fixed and vary x. Then for a given point x in R, there will correspond a value of m (x, $\in$). In this way, we shall get a set of values of m(x, $\in$). This set of values of m(x, $\in$) may or may not have an upper bound. If this set has an upper bound, say M, then for every point x in R, we have

$$n \geq M \Rightarrow |f_n(x) - f(x)| < \in.$$

In such a case, we say that the sequence $\{f_n\}$ converge uniformly to f on X.

Definition. A sequence $\{f_n\}$ of functions is said to converge uniformly on R to a function f if for every $\in > 0$, there can be found a positive integer m such that

$$n \geq m \Rightarrow |f_n(x) - f(x)| < \in$$

for all $x \in R$.

Remark. Observe that the convergence of a sequence $\{f_n(x)\}$ at every point (i.e., point-wise convergence) does not necessarily ensure its uniform convergence on R. A sequence of functions may be convergent at every point of R and yet may not be uniformly convergent on R. For example, consider the sequence $\{f_n\}$ defined on [0, 1] as follows by $f_n(x) = x^n$.

Here, we have $\lim_{n\to\infty} x^n = 0$ if $0 \leq x < 1$

and $\lim_{n\to\infty} x^n = 1$ if $x = 1$.

Thus the limit function f is defined by

$$f(x)\begin{cases}0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad x = 1.\end{cases}$$

The function f_n therefore has a definite limit for every value of x in [0, 1] as $n \to \infty$ and consequently the sequence $\{f_n(x)\}$ converges for every $x \in [0, 1]$.

to see whether the convergence is uniform, we consider the interval [0, 1]. Let $\in > 0$ be given. Then

$$|f_n(x) - f(x)| < \in \Rightarrow |x^n - 0| < \in \Rightarrow x_n < \in \Rightarrow \frac{1}{x^n} > \frac{1}{\in}$$

$$\Rightarrow n\log\frac{1}{x} > \log\frac{1}{\in} \Rightarrow n > \frac{\log(1/\in)}{\log(1/x)} \qquad ...(1)$$

Thus when $x \neq 1$, $m(x, \in)$ is any integer greater than

$\log(1/\in)/\log(1/x)$.

In particular $m(x, \in) = 1$ when $x = 0$.

Now as x, starting from 0, increases and approaches 1, it is evident from (1) that $n \to \infty$ and so it is not possible to determine a positive integer m such that

$$n \ge m \Rightarrow |f_n(x - f(x))| < \in$$

for all $x \in [0, 1[$.

Thus $\{f_n\}$ is not uniformly convergent in [0, 1[.

If, however, we consider the interval $0 \le x \le k$, where $0 < k < 1$, we see that the greatest value of $\log(1/\in)/\log(1/x)$ is $\log(1/\in).\log(1/k)$ so that if we take m equal to any positive integer greater than this greatest value, we have

$$n \ge m \Rightarrow |f_n(x) - f(x)| < \in$$

for all $x \in [0, k]$

Thus $\{fn(x)\}$ converges unformly on [0, k].

Uniform Convergence and Differentiation.

Theorem. Let $\{f_n\}$ be a sequence of the real valued functions defined on [a, b] such that

(i) f_n is differentiable on [a, b] for n = 1, 2, 3, ...,

(ii) The sequence $\{f_n(c)\}$ converges for some point c of [a, b],

(iii) The sequence $\{f'_n\}$ converges uniformly on [a, b].

Then the sequence $\{f_n\}$ converges uniformly to a differentiable limit f and

$$\lim_{x\to\infty} f_n'(x) = f'(x) \qquad (a \le c \le b).$$

Proof. Let $\in > 0$ be given. Then by the convergence of $\{f_n(c)\}$ and by the uniform convergence of $\{f_n'\}$ on [a, b], there exists a positive integer m such that for all $n \ge m$, $p \ge m$,

we have $$|f_n(c) - f_p(c)| < \frac{\in}{2} \qquad ...(1)$$

and $$|f_n'(x) - f_p'(x)| < \frac{\in}{2(b-a)} \quad (a \le x \le b). \qquad ...(2)$$

Applying the mean value theorem of differential calculus to the function $f_n - f_p$, we have

$$[f_n(x) - f_p(x)] - [f_n(y) - f_p(y)] = (x - y)\,[f_n'(\xi) - f_p'(\xi)]$$

For any x and y in [a, b] and for some ξ between x and y provided $n \ge m$, $p \ge m$. Hence

$$|f_n(x) - f_p(x) - f_n(y) + f_p(y)| = |x - y|\,|f_n'(\xi) - f_p'(\xi)|$$

$$< \frac{|x-y|\in}{2(b-a)} \text{ by (2)} \qquad ...(3)$$

$$< \frac{\in}{2}\left[\because |x-y| \le (b-a)\right] \qquad ... (4)$$

for all $n, p \ge m$ and all $x, y \in [a, b]$. Now

$$|f_n(x) - f_p(x)| = |f_n(x) - f_p(x) - f_n(c) + f_p(c) + f_n(c) - f_p(c)|$$

$$\le |f_n(x) - f_p(x) - f_n(c) + f_p(c)| + |f_n(c) - f_p(c)$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in \qquad \text{by (1) and (4),}$$

for all $n, p \ge m$ and for all $x \in [a, b]$. Thus we have shown that given $\in > 0$, there exists a positive integer m such that

$$n \ge m,\ p \ge m,\ x \in [a, b] \quad \Rightarrow \quad |f_n(x) - f_p(x)| < \in.$$

It follows that the sequence $\{f_n\}$ converges uniformly to a function f and so

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (a \le x \le b).$$

This proves the first result.

Now for an arbitrary but for the moment a fixed $x \in [a, b]$, define

$$F_n(y) = \frac{f_n(y) - f_n(x)}{y - x} \qquad F(y) = \frac{f(y) - f(x)}{y - x} \quad , \quad , \qquad \text{...(5)}$$

for $a \le y \le b$, $y \ne x$. Then

$$\lim_{y\to x} F_n(y) = \lim_{y\to x} \frac{f_n(y) - f_n(x)}{y - x} = f_n'(x) \qquad \text{...(6)}$$

for n = 1, 2, 3, ...

Now for $n \ge m$, $p \ge m$, we have

$$|F_n(y) - F_p(y)| = \left|\frac{f_n(y) - f_n(x) + f_p(y) - f_p(x)}{y - x}\right|$$

$$< \frac{\in}{2(b-a)} \qquad \text{by (3).}$$

It follows that $\{F_n\}$ converges uniformly for $y \ne x$. Since $\{f_n\}$ converges to f, we conclude from (5) that

$$\lim_{n\to\infty} F_n(y) = \lim_{n\to\infty} \frac{f_n(y) - f_n(x)}{y - x} = \frac{f(y) - f(x)}{y - x} = F(y) \qquad \text{...(7)}$$

Uniformly for $a \le y \le b$, $y \ne x$.

$$\lim_{y\to x} F(y) = \lim_{n\to\infty} f_n'(x)$$

or $$\lim_{y\to x} \frac{f(y) - f(x)}{y - x} = \lim_{n\to\infty} f_n'(x) \text{ by (5)}$$

or $$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad \text{...(8)}$$

for every $x \in [a, b]$.

The theorem is thus completely established.

Term by term differentiation.

Cor. Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued differentiable functions on [a, b] such that $\sum_{n=1}^{\infty} u_n(c)$ converges for some point c of [a, b] and $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on [a, b]. Then the series $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on [a, b] to a differentiable sum function f and

$$, f'(x) = \lim_{n\to\infty} \sum_{m=1}^{n} u'_m(x) \quad (a \le x \le b).$$

In other words, if $a \le x \le b$, then

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty}\left[\frac{d}{dx} u_n(x)\right]$$

Proof. Let $f_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$.

Then $f_n'(x) = u_1'(x) + u_2'(x) + ... + u_n'(x)$

[$\because$ The differential coefficient of the sum of a finite number of differentiable functions is equal to the sum of their differential coefficients].

Hence the series $\sum_{n=1}^{\infty} u_n(x)$ and $\sum_{n=1}^{\infty} u'_n(x)$ are respectively equivalent to the sequences $\{f_n\}$ and $\{f_n'\}$. Now proceed as in the above theorem.

Theorem . Let $\{f_n\}$ be a sequence of real valued functions defined on [a, b] such that

(i) f_n is differentiable on [a, b] for n = 1, 2, 3, ...

(ii) the sequence $\{f_n\}$ converges to f on [a, b],

(iii) the sequence $\{f_n'\}$ converges uniformly on [a, b] to g,

(iv) each f_n' is continuous on [a, b].

Then $g(x) = f'(x)$ $(a \le x \le b)$. That is,

$$\lim_{n\to\infty} f_n'(x) = f'(x) \quad (a \le x \le b).$$

Proof. Since $\{f_n'\}$ is a uniformly convergent sequence of continuous functions, it follows that g is continuous on [a, b]. Moreover $\{f_n'\}$ converges uniformly to g on [a, x] where x is any point of [a, b]. Then we have

$$\lim_{n\to\infty}\int_a^x f_n'(t)\,dt = \int_a^x g(t)\,dt \qquad ...(1)$$

But by the fundamental theorem of integral calculus, we have

$$\int_a^x f_n'(t)\,dt = f_n(x) - f_n(a).$$

Also by hypothesis,

$\lim_{n\to\infty} f_n(x) = f(x)$ and $\lim_{n\to\infty} f_n(a) = f(a)$.

Hence (1) gives

$$f(x) - f(a) = \int_a^x g(t)\,dt \quad (a \le x \le b).$$

It follows

$$f'(x) = g(x) \qquad (a \le x \le b)$$

or $\quad f'(x) = \lim_{n\to\infty} f_n'(x)$

Ex. Consider the series $\sum \frac{(-1)^{n-1}}{(n+x^2)}$ for uniform convergence for all values of x.

Sol. Let $u_n = (-1)^{n-1}$, $v_n(x) = \frac{1}{n+x^2}$.

Since $f_n(x) = \sum_{r=1}^{n} u_r = 0$ or 1 according as n is even or odd, $f_n(x)$ is bounded for all n.

Also $v_n(x)$ is a positive monotonic decreasing sequence converging to zero for all real values of x.

Hence the given series is uniformly convergent for all real values of x.

Ex. If $f(x) = \sum_1^\infty \frac{1}{n^3 + n^4x^2}$, then find its differential coefficient

(A) $-2x\sum_1^\infty \frac{1}{n^2(1+nx^2)^2}$ (B) $2x\sum_1^\infty \frac{1}{n^2(1+nx^2)^2}$ (C) $\sum_1^\infty \frac{1}{n^2(1+nx^2)^2}$ (D) $\sum_1^\infty \frac{-1}{n^2(1+nx^2)^2}$

Sol. Here $\quad u_n(x) = \frac{1}{n^3 + n^4x^2}$

and $\quad u_n'(x) = \dfrac{2x}{n^2\left(1+nx^2\right)^2}$.

Now $\quad u_n'$ (x) is maximum when $\dfrac{du'_n(x)}{dx} = 0$

i.e. $\quad (1 + nx^2)^2 - 4nx^2(1 + nx^2) = 0$

or $\quad 1 - 3nx^2 = 0 \quad$ or $\quad x = \dfrac{1}{\sqrt{(3n)}}$.

$\therefore \quad$ Max. $|u'_n(x)| = \dfrac{2}{\sqrt{3n^{5/2}\left(1+\frac{1}{3}\right)^2}} = \dfrac{3\sqrt{3}}{8n^{5/2}}$.

Then $\quad |u'_n(x)| < \dfrac{1}{n^{5/2}}$ for all values of x.

But $\sum \dfrac{1}{n^{5/2}}$ is convergent.

Hence by Weierstrass's M-test, the series $\Sigma u_n'$ is uniformly convergent for all real values of x. The term by term differentiation is therefore justified.

Hence . $f'(x) = \sum_{n=1}^{\infty} u_n'(x) = -2x\sum_{1}^{\infty}\dfrac{1}{n^2\left(1+nx^2\right)^2}$